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# Contractions: Nijenhuis and Saletan tensors for general algebraic structures

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**Abstract**

We study generalizations in many directions of the contraction procedure for Lie algebras introduced by Saletan. We consider products of an arbitrary nature, not necessarily Lie brackets, and we generalize to infinite dimension, considering a modification of the approach by Nijenhuis tensors to bilinear operations on sections of finite-dimensional vector bundles. We apply our general procedure to Lie algebras, Lie algebroids and Poisson brackets. We also present results on contractions of $n$-ary products and coproducts.

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## 1. Introduction

For a general (real) topological algebra, i.e. a topological vector space $\mathcal{A}$ over $\mathbb{R}$ (but other topological fields, like $\mathbb{C}$, can be considered in a similar way as well) with a continuous bilinear operation

$$\mu : \mathcal{A} \times \mathcal{A} \to \mathcal{A} \qquad (X, Y) \mapsto X * Y \tag{1}$$

one considers contraction procedures as follows.

If $U(\lambda) : \mathcal{A} \to \mathcal{A}$ is a family of linear morphisms which continuously depends on the parameter $\lambda \in \mathbb{R}$ from a neighbourhood $\mathcal{U}$ of 0 and $U(\lambda)$ are invertible for $\lambda \in \mathcal{U} \setminus \{0\}$, then we can consider the continuous family of products $X *^{\lambda} Y$ defined by

$$X *^{\lambda} Y = U(\lambda)^{-1}(U(\lambda)(X) * U(\lambda)(Y)) \tag{2}$$

for $\lambda \in \mathcal{U} \setminus \{0\}$. All of these products are isomorphic by definition, since

$$U(\lambda)(X *^{\lambda} Y) = U(\lambda)(X) * U(\lambda)(Y) \tag{3}$$

and if $N = U(0)$ is invertible, then clearly

$$\lim_{\lambda\to 0} X *^{\lambda} Y = N^{-1}(N(X) * N(Y)). \tag{4}$$

But sometimes, the limit $\lim_{\lambda\to 0} X *^{\lambda} Y$ may exist for all $X, Y \in \mathcal{A}$ even if $N$ is not invertible and (4) does not make sense. We say then that $\lim_{\lambda\to 0} X *^{\lambda} Y$ is a *contraction* of the product $X * Y$. Of course, the problem of existence and the form of the contracted product depends heavily on the family $U(\lambda)$. In [Sa] this problem has been solved for linear families $U(\lambda) = \lambda I + N$ and $\mathcal{A}$—a finite-dimensional Lie algebra.

Here we study generalizations in various directions of the contraction procedure introduced by Saletan [Sa]. First of all, we consider products of an arbitrary nature, not necessarily Lie brackets. Second, we generalize to infinite dimension, considering a modification of the approach by Nijenhuis tensors to bilinear operations on sections of finite-dimensional vector bundles. The motivation stems from physics, since infinite-dimensional algebras of sections of some bundles arise frequently as models both in classical and quantum physics. In particular, we were confronted with this problem within the framework of quantum bihamiltonian systems [CGM]. According to Dirac [Di], a 'quantum Poisson bracket' necessarily arises from the associative product on the space of operators. Similarly, by Ado's theorem, any finite-dimensional Lie algebra arises as an algebra of matrices. It is therefore quite natural to investigate contractions of associative algebras along with contractions of Lie algebras and their generalizations to Lie algebroids. We concentrate mainly on smooth sections, but this particular choice plays no definite role in our approach.

The paper is organized as follows. In the next section we present the general scheme we are working with and the main result (theorem 2) on contractions for algebras of sections of vector bundles. We make some remarks on contractions with respect to more general linear families $U(\lambda) = \lambda A + N$. Section 3 is devoted to examples and section 4 to more detailed studies of hierarchies of contractions. We develop an algebraic technique which allows us to produce much simpler proofs of facts about hierarchies than those available in the literature. In section 5 we comment on the behaviour of algebraic properties under contractions. Contractions of Lie algebras and Lie algebroids, as particular cases of our general procedure, are studied in sections 6 and 7. In section 8 we use our knowledge on contractions of Lie algebroids to define contractions of Poisson structures. The approach is very natural and leads to structures very similar (but slightly different) to those which are known under the name of Poisson–Nijenhuis structures (cf [MM, KSM]). We end with observations on contractions of $n$-ary products and coproducts.

## 2. Linear contractions of products on sections of vector bundles

Let us assume that $E$ is a smooth vector bundle with fibres of dimension $n_0$, over a smooth manifold $M$. Denote by $*$ a bilinear operation $\mu : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$,

$$\mu : (X, Y) \in \mathcal{A} \otimes \mathcal{A} \mapsto X * Y \in \mathcal{A}$$

on the space of smooth sections of $E$ which is, at least pointwise, continuous. In practice, we shall deal with local products, therefore being defined by bilinear differential operators. We shall use both notations for the product according to which is more convenient when treating particular cases.

Let $N : E \to E$ be a smooth vector bundle morphism over $\mathrm{id}_M$. One refers also to $N$ as a (1, 1)-tensor field, i.e. a section of $E^* \otimes E$. Since $N_p : E_p \to E_p$ is a morphism of the finite-dimensional vector space $E_p$, where $E_p$ denotes the fibre over the point $p \in M$, we have the Riesz decomposition $E_p = E_p^1 \oplus E_p^2$ into invariant subspaces of $N_p$ in such a way that $N_p$ is invertible on $E_p^1$ and nilpotent of order $q$ on $E_p^2$, i.e. $N^q(X_p) = 0$, for $X_p \in E_p^2$. One can take $E_p^1 = \widetilde{N}_p(E_p)$, $E_p^2 = \ker \widetilde{N}_p$, where $\widetilde{N}_p = (N_p)^{n_0}$, with $n_0 = \dim E_p$. In this way we get the decomposition $E = E^1 \oplus E^2$ of the vector bundle $E$ into two supplementary generalized distributions. Note that the dimension of $E_p^1$ may vary from point to point. Nevertheless, $E^1$ is a smooth distribution, i.e. it is generated locally by a finite number of smooth sections of $E$. Indeed, if $\{e_1, \ldots, e_{n_0}\}$ is a local basis of smooth sections of $E$, then $\{\widetilde{N}(e_1), \ldots, \widetilde{N}(e_{n_0})\}$ is a set of local smooth sections generating locally $E^1$.

**Theorem 1.** *The (generalized) distribution $E^2$ is smooth if and only if it is regular, i.e. of constant rank:* $\dim E_p^2 = $ *constant.*

**Proof.** Since the rank of a smooth distribution is semi-continuous from above

$$\lim_{p\to p_0} \inf \dim E_p^2 \geqslant \dim E_{p_0}^2 \tag{5}$$

and the complementary distribution $E^1$ is smooth, so that

$$\lim_{p\to p_0} \sup \dim E_p^2 \leqslant \dim E_{p_0}^2 \tag{6}$$

we see that both conditions (5) and (6) are satisfied if and only if $E^2$ is of constant rank.

Conversely, if $E^2$ is of constant rank, say $n_0 - l$, take a basis $\{e_1, \ldots, e_{n_0}\}$ of smooth local sections of $E$ such that the elements of $\{\widetilde{N}(e_1), \ldots, \widetilde{N}(e_l)\}$ span $E_p^1$. Then $\{\widetilde{N}(e_1), \ldots, \widetilde{N}(e_l)\}$ is a basis of local sections of $E^1$ near $p \in M$. Write

$$\widetilde{N}(e_i) = \sum_{j=1}^{l} f_{ij}\, \widetilde{N}(e_j).$$

Then the functions $f_{ij}$ are smooth and the smooth sections

$$\widetilde{e}_i = e_i - \sum_{j=1}^{l} f_{ij}\, e_j \qquad i = l+1, \ldots, n_0$$

span $E^2$ locally. Indeed, $\widetilde{N}(\widetilde{e}_i) = 0$ and the elements $\widetilde{e}_i$, for $i = l+1, \ldots, n_0$, are linearly independent. $\square$

Note that in general none of the distributions $E^1$ and $E^2$ has to be 'involutive' in the sense that smooth sections of $E^1$ (respectively, $E^2$) are closed with respect to the composition law $*$.

Consider now a new (1, 1)-tensor $U(\lambda) = \lambda I + N$ depending on a real parameter $\lambda$. Since the spectrum of $N$ is finite and depends continuously on $p$, in a sufficiently small neighbourhood of $p$ all of $U(\lambda)_p$ are invertible for sufficiently small $\lambda$, but $\lambda \neq 0$. Thus, we can locally define, for $\lambda \neq 0$, a new operation

$$\begin{aligned} X *_N^\lambda Y &= U(\lambda)^{-1}(U(\lambda)(X) * U(\lambda)(Y)) \\ &= U(\lambda)^{-1}((\lambda X + N(X)) * (\lambda Y + N(Y))) \\ &= U(\lambda)^{-1}(\lambda^2 X * Y + \lambda (N(X) * Y + X * N(Y)) + N(X) * N(Y)). \end{aligned} \tag{7}$$

We would like to find conditions ensuring that the limit

$$X *_N Y = \lim_{\lambda\to 0} X *_N^\lambda Y$$

exists for all $X, Y \in \mathcal{A}$ and find the corresponding contraction $X *_N Y$.

Using the identity $U(\lambda)^{-1}(\lambda I + N) = I$, i.e.

$$U(\lambda)^{-1}(\lambda X) = X - U(\lambda)^{-1}N(X)$$

we obtain from (7) that

$$\begin{aligned} X *_N^\lambda Y = \lambda X * Y + (N(X) * Y + X * N(Y) - N(X * Y)) \\ +U(\lambda)^{-1}(N(X) * N(Y) - N(N(X) * Y + X * N(Y) - N(X * Y))). \end{aligned} \tag{8}$$

Denoting

$$\delta_N \mu(X, Y) = X \widetilde{*}_N Y = N(X) * Y + X * N(Y) - N(X * Y)$$

and by $T_N \mu(X, Y)$ the *Nijenhuis torsion* of $N$:

$$T_N \mu(X, Y) = N(X) * N(Y) - N(X \widetilde{*}_N Y)$$

we can rewrite (8) in the form

$$X *_N^\lambda Y = \lambda X * Y + X \widetilde{*}_N Y + U(\lambda)^{-1} T_N \mu(X, Y).$$

Hence, the limit

$$\lim_{\lambda \to 0} X *_N^\lambda Y$$

exists if and only if

$$\lim_{\lambda \to 0} U(\lambda)^{-1} T_N \mu(X, Y) \qquad \text{exists for every} \quad X, Y \in \mathcal{A}. \tag{9}$$

Denote by $\mathcal{A}^1$, $\mathcal{A}^2$, the spaces of smooth sections of $E^1$ and $E^2$, respectively. Of course, in general $\mathcal{A} \neq \mathcal{A}^1 \oplus \mathcal{A}^2$. We may have $\mathcal{A}^2 = \{0\}$ even in the case $E^2 \neq \{0\}$. Since $E^1$ and $E^2$ are invariant distributions of $U(\lambda)$, and hence of $U(\lambda)^{-1}$, the existence of the limit (9) may be checked separately on the corresponding parts of $T_N \mu$. On $E^2$ the tensor $N$ is nilpotent, so for $X_p \in E_p^2$,

$$(\lambda I + N)_p^{-1}(X_p) = (\lambda(I - (-N/\lambda)))_p^{-1}(X_p) = \frac{1}{\lambda} \sum_{n=0}^{\infty} \left(-\frac{1}{\lambda}\right)^n N_p^n(X_p) \tag{10}$$

where the sum is, in fact, finite, and

$$\lim_{\lambda \to 0} \sum_{n=0}^{q-1} \frac{(-1)^n}{\lambda^{n+1}} N_p^n(X_p)$$

exists if and only if $X_p = 0$. Thus, a necessary condition for existence of the limit (9) is that $T_N \mu(X, Y) \in \mathcal{A}^1$ for every $X, Y \in \mathcal{A}$.

Since on $E^1$ the tensor $N$ is invertible, we have clearly

$$\lim_{\lambda \to 0} (\lambda I + N)^{-1} = N^{-1}$$

on $E^1$, so that, assuming $T_N \mu(X, Y) \in \mathcal{A}^1$,

$$\lim_{\lambda \to 0} U(\lambda)^{-1} T_N \mu(X, Y) = N^{-1} T_N \mu(X, Y) = \tau_N \mu(X, Y).$$

Here $\tau_N \mu(X, Y) = N^{-1} T_N \mu(X, Y)$ is the unique section of $E^1$ determined by the condition

$$N(\tau_N \mu(X, Y)) = T_N \mu(X, Y).$$

In order to obtain a new product on $\mathcal{A}$ we have to assume that $\tau_N \mu(X, Y)$ is smooth, which is *a priori* not automatic, even if we have $T_N \mu(X, Y) \in \mathcal{A}^1$. Note that if $N$ is regular, i.e. $E^1$ is of constant dimension, then, as we shall show in theorem 3, $N(\mathcal{A}^1) = \mathcal{A}^1$ and $\tau_N \mu(X, Y)$ is smooth automatically.

Let us summarize the above as follows:

**Theorem 2.** *Let $\mu : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ be a pointwise continuous bilinear product of smooth sections of a vector bundle $E$ over a manifold $M$ (we will write also $X * Y$ instead of $\mu(X, Y)$) and let $N : E \to E$ be a smooth $(1,1)$-tensor. Denote by $U(\lambda) = \lambda I + N$ a deformation of $N$, by $E = E^1 \oplus E^2$ the Riesz decomposition of $E$ relative to $N$, and by $\mathcal{A}^1$ the set of smooth sections of $E^1$. Then, the limit*

$$\lim_{\lambda \to 0} U(\lambda)^{-1}(U(\lambda)(X) * U(\lambda)(Y))$$

*exists for all $X, Y \in \mathcal{A}$ and defines a new (contracted) bilinear operation*

$$D_N\mu(X, Y) = X *_N Y$$

*on $\mathcal{A}$ if and only if the Nijenhuis torsion*

$$T_N\mu(X, Y) = N(X) * N(Y) - N(N(X) * Y + X * N(Y)) + N^2(X * Y)$$

*takes values in $N(\mathcal{A}^1)$. If this is the case, then*

$$X *_N Y = X \,\widetilde{*}_N\, Y + \tau_N\mu(X, Y) \tag{11}$$

*where $X \,\widetilde{*}_N\, Y$ is a new bilinear operation $\delta_N\mu$ on $\mathcal{A}$ defined by*

$$\delta_N\mu = X \,\widetilde{*}_N\, Y = N(X) * Y + X * N(Y) - N(X * Y)$$

*and $\tau_N\mu(X, Y) = N^{-1}T_N\mu(X, Y)$ is the unique section of $\mathcal{A}^1$ such that*

$$N(\tau_N\mu(X, Y)) = T_N\mu(X, Y).$$

*Moreover, $N$ constitutes a homomorphism of $(\mathcal{A}, \mu_N)$ into $(\mathcal{A}, \mu)$:*

$$N(X *_N Y) = N(X) * N(Y).$$

**Remark.** Let us note that our procedure is not just applying the finite-dimensional linear one to every fibre, since the operation $*$ need not act fibrewise. Also, this is not direct application to infinite-dimensional algebra $\mathcal{A}$, since we have not, in general, the Riesz decomposition $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ with respect to $N$. On the other hand, the whole procedure can be applied directly to infinite-dimensional cases for which we are given the Riesz decomposition of $N$.

**Definition 1.** *The tensor $N$ satisfying the assumptions of theorem 2, i.e. such that $T_N\mu$ takes values in $N(\mathcal{A}^1)$, will be called* a Saletan tensor. *If $N^k$ is a Saletan tensor for every $k = 1, 2, 3, \ldots$, then $N$ will be called a* strong Saletan tensor. *In the case $T_N\mu = 0$ we shall call $N$ a* Nijenhuis tensor.

**Remark.** It is obvious from the proof of theorem 2 that Nijenhuis tensors define contractions even in the case of infinite-dimensional algebras $\mathcal{A}$ without any assumption that $\mathcal{A}$ consists of sections of a finite-dimensional vector bundle. Indeed, with $T_N\mu = 0$ we have no obstructions, the Riesz decomposition is irrelevant, and $X *_N Y = X \,\tilde{*}_N\, Y$. Of course,

$$\text{(Nijenhuis)} \Rightarrow \text{(strong Saletan)} \Rightarrow \text{(Saletan)}.$$

We shall call $N$ *regular*, if $E^1$ (hence also $E^2$) is of constant rank. This is always the case when $E$ is a bundle over a single point, i.e. $E = \mathcal{A}$.

**Theorem 3.** *In the regular case, i.e. when $E^1$ is of constant rank, $N(\mathcal{A}^1) = \mathcal{A}^1$, so that $N$ is a Saletan tensor if and only if $T_N\mu(X, Y)$ takes values in $\mathcal{A}^1$.*

**Remark.** We shall prove a stronger result in theorem 6.

**Proof.** Indeed, according to theorem 1, both $E^1$ and $E^2$ are smooth distributions. Locally we have a basis of smooth sections of $E^1$, and $N$ acts on this basis as invertible matrix of smooth functions. Indeed, since regularity of $E^1$ implies that there is a local basis $\{e_1, \ldots, e_l\}$ of sections of $E^1$, in this basis $N$ acts simply as invertible matrix of smooth functions $(f_{ij})$, so for $X = \sum g_i\, e_i$,

$$N\left(\sum_{i=1}^{l} h_i\, e_i\right) = X$$

where the smooth functions $h_i$ are defined by

$$\sum_{j=1}^{l} f_{ij}\, h_j = g_i.$$

Hence, $N^{-1}(X)$ is locally, thus globally, smooth section of $E^1$ for any smooth section $X$ of $E^1$. □

**Remark.** For a fibre bundle over a single point theorem 2 gives exactly the Saletan result [Sa] in the case where $\mu$ is a Lie bracket. Saletan writes $X *_N Y$ in the form

$$X *_N Y = (X\, \widetilde{*}_N\, Y)_2 + N^{-1}\left((N(X) * N(Y))_1\right) \tag{12}$$

where $X = X_1 + X_2$ is the decomposition of $X \in \mathcal{A}$ into sections of $E^1$ and $E^2$. Of course, equation (12) is formally the same as (11) for the decomposition into sections of $E^1$ and $E^2$. However, in general the summands of the right-hand side of (12) are not smooth, while the decomposition (11) is into smooth parts. In the regular case both formulae coincide.

**Theorem 4.**

(a) *Theorem 2 remains valid when we consider the family $U(\lambda)$ in a slightly more general form: $U(\lambda) = \lambda\, I + f(\lambda)\, N$, where $f$ is continuous and $f(0) = 1$.*

(b) *If we consider instead of $U(\lambda)$ the family $U_1(\lambda) = \lambda A + N$, then the contraction procedure for $U_1(\lambda)$ and the product $*$ is equivalent to the contraction procedure of the above type for a new $N$ and a new product. In particular, if $A$ is invertible, we get our standard contraction for $A^{-1}N$ and the product $X *_A Y = A^{-1}(A(X) * A(Y))$. In other words, the contraction procedure for the family $U_1(\lambda) = \lambda\, A + N$ can be reduced to the contraction described in theorem 2.*

**Proof.**

(a) Let us write $U(\lambda) = \frac{U_1(\varepsilon)}{f(\lambda)}$, where $U_1(\varepsilon) = \varepsilon\, I + N$ and $\varepsilon = \frac{\lambda}{f(\lambda)}$, so that $\lambda \to 0$ is equivalent to $\varepsilon \to 0$. Since

$$U(\lambda)^{-1}(U(\lambda)(X) * U(\lambda)(Y)) = \frac{1}{f(\lambda)}\, U_1(\varepsilon)^{-1}(U_1(\varepsilon)(X) * U_1(\varepsilon)(Y)) \tag{13}$$

and $\lim_{\lambda\to 0} f(\lambda) = 1$, both contraction procedures are equivalent.

(b) Assume first that $A$ is invertible. Since $\lambda A+N = A(\lambda I+A^{-1}N)$, we can use theorem 2 for $N := A^{-1}N$ and the product $*_A$. In fact, we can skip the assumption that $A$ is invertible. For, take $\lambda_0$ for which $A + \lambda_0\mathfrak{N}$ is invertible. Then, we write

$$U(\lambda) = \lambda A + N = (A + \lambda_0 N)\,(\lambda I + (1 - \lambda\,\lambda_0)(A + \lambda_0 N)^{-1}N) \qquad (14)$$

and we can proceed as before and using (a) of the theorem. □

## 3. Examples

Many interesting physical applications are based on the idea of contraction by İnönü and Wigner [IW]. We will call a smooth distribution $E^1$ in the vector bundle $E$ *involutive* if the space $\mathcal{A}^1$ of sections of $E^1$ is closed with respect to the product $*$, i.e. $\mathcal{A}^1$ is a subalgebra of $\mathcal{A}$.

**Theorem 5.** *Let $E^1$ be a smooth regular and involutive distribution in $E$. Take $E^2$ to be any supplementary smooth distribution and let $N = P_{E^1}$ be the projection on $E^1$ along $E^2$. Then $N$ is a Saletan tensor which is Nijenhuis if and only if $E^2$ is also involutive. The contracted product reads*

$$X *_N Y = X_1 * Y_1 + (X_1 * Y_2 + X_2 * Y_1)_2 \qquad (15)$$

*where $X = X_1 + X_2$, etc, is the decomposition with respect to the splitting $E = E^1 \oplus E^2$.*

**Proof.** It is obvious that the Nijenhuis tensor $T_N(X, Y) = N(X) * N(Y) - N(X \,\widetilde{*}_N\, Y)$ takes values in $\mathcal{A}^1$, since $E^1$ is involutive. Due to regularity, the corresponding contraction exists (theorem 3). It is easy to see that

$$X \,\widetilde{*}_N\, Y = X_1 * Y_1 + (X_1 * Y_2 + X_2 * Y_1)_2 - (X_2 * Y_2)_1. \qquad (16)$$

Hence, $T_N(X, Y) = \tau_N(X, Y) = (X_2 * Y_2)_1$, so that $N$ is a Nijenhuis tensor if and only if $E^2$ is also involutive. Finally,

$$X *_N Y = X \,\widetilde{*}_N\, Y + \tau_N(X, Y) = X_1 * Y_1 + (X_1 * Y_2 + X_2 * Y_1)_2. \qquad (17)$$

□

**Example 1.** Consider a manifold $M$ with two foliations $\mathcal{F}_1, \mathcal{F}_2$ corresponding to a splitting into complementary distributions $TM = E^1 \oplus E^2$. The projection $N$ of $TM$ onto $E^1$ along $E^2$ is a Nijenhuis tensor (theorem 5). The contracted bracket is trivial for two vector fields which are tangent to $\mathcal{F}_2$, it is the standard one for two vector fields which are tangent to $\mathcal{F}_1$ and it is the projection onto $E^2$ of the standard bracket of two vector fields of which one belongs to $\mathcal{F}_1$ and the second to $\mathcal{F}_2$.

**Example 2.** Let $E$ be just one-dimensional trivial bundle over $\mathbb{R}$, i.e. $\mathcal{A} = C^\infty(\mathbb{R})$. Take $f * g = f' g'$ and $N = \varphi I$, where $\varphi \in \mathcal{A}$. Then $E^1_p = T_p\mathbb{R}$ if $\varphi(p) \neq 0$ and $E^1_p = \{0\}$ otherwise, so the distribution need not to be regular. We have

$$f \,\widetilde{*}_N\, g = \varphi\, f' g' + \varphi' (f' g + f g')$$

and $T_N\mu(f, g) = \varphi' \varphi' f g$. For instance, if $\varphi(p) = p^2$ (non-regular case), then

$$T_N\mu(f, g) = 4\varphi\, f g$$

i.e. $N$ is not Nijenhuis but satisfies the assumptions of the theorem. We obtain

$$f *_N g = f \,\widetilde{*}_N\, g + N^{-1}(T_N\mu(f, g)) = \varphi\, f' g' + \varphi'(f'g + fg') + 4 f g.$$

**Example 3.** It is easy to see that if $*$ is an associative product, the multiplication by any $K \in \mathcal{A}$:

$$N_K : \mathcal{A} \to \mathcal{A} \qquad N_K(X) = KX$$

is a Nijenhuis tensor. In view of remark 4, the corresponding contraction yields

$$X *_{N_K} Y = X * K * Y.$$

This product has been recently used as an alternative product of operators in quantum mechanics in connection with deformed oscillators [MMSZ], taking up an old idea of Wigner [Wi].

**Example 4.** Another alternative product for quantum mechanics can be constructed as a contraction as follows (cf [CGM]). Now let the algebra $\mathcal{A}$ be the algebra of $n \times n$ matrices, $n = 1, 2, \ldots, \infty$. In the case $n = \infty$ we consider infinite matrices concentrated on the diagonal, i.e. matrices which are null outside a strip of the diagonal. The algebra $\mathcal{A}$ then represents unbounded operators on a Hilbert space $\mathcal{H}$ with a common dense domain. We choose $\mathcal{A}_1$ to be a subalgebra of upper-triangular matrices and for $\mathcal{A}_2$ we take the supplementary algebra of strict lower-triangular matrices. Then, the mapping

$$N_\alpha(A) = (1-\alpha)A_1 + \alpha A \tag{18}$$

is a Nijenhuis tensor on $\mathcal{A}$ for every $\alpha \in \mathbb{C}$. For example, for $n = 2$, the new associative matrix multiplication has the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa' + \alpha bc' & ab' + bd' \\ ca' + dc' & dd' + \alpha cb' \end{pmatrix}. \tag{19}$$

Note that the unit matrix $I$ remains the unit for this new product and that inner derivations given by diagonal matrices are the same for both products.

Since the corresponding deformed associative products $*_{N_\alpha}$ give all the same results if one of the factors is a diagonal matrix, in the infinite case $n = \infty$ the Hamiltonian $H$ for the harmonic oscillator, $H\,|e_n\rangle = n\,|e_n\rangle$, describes the same motion for all deformed brackets. This time, however, $a^\dagger *_{N_\alpha} a = \alpha H$, so $a^\dagger$ and $a$ commute for $\alpha = 0$.

## 4. Hierarchy of contractions

Let us have a look at the process of constructing contracted products in a more systematic way. Denote the linear space of all bilinear products on $\mathcal{A}$ by $\mathcal{B}$, the linear subspace of all bilinear products $\mu$ such that $\mu(X, Y) \in N^k(\mathcal{A}^1)$ by $\mathcal{B}^1_k$. Note that the distribution $E^1$ associated with the (1, 1)-tensor field $N$ ($N$ will be fixed) is the same for all positive powers of $N$. Let $A_N, B_N, C_N : \mathcal{B} \to \mathcal{B}$ be given by

$$(A_N\mu)(X, Y) = N(\mu(X, Y)) \tag{20}$$

$$(B_N\mu)(X, Y) = \mu(N(X), Y) \tag{21}$$

$$(C_N\mu)(X, Y) = \mu(X, N(Y)). \tag{22}$$

It is easy to see that $A_N, B_N, C_N$ generate a commutative algebra of linear operators on $\mathcal{B}$ for which $\mathcal{B}^1_k$ are invariant subspaces. Moreover, $A_{N^k} = (A_N)^k$, etc. Observe that for the derived product,

$$(\delta_N\mu)(X, Y) = \mu(N(X), Y) + \mu(X, N(Y)) - N(\mu(X, Y))$$

we can write

$$\delta_N = B_N + C_N - A_N$$

and for the Nijenhuis torsion,

$$(T_N\mu)(X, Y) = \mu(N(X), N(Y)) - N(\delta_N\mu(X, Y))$$

we can write

$$T_N = B_N C_N - A_N\delta_N = (A_N - B_N)(A_N - C_N).$$

The contracted product $D_N\mu$ is defined via the formula

$$\mu_N = D_N\mu = \delta_N\mu + \tau_N\mu$$

where $\tau_N\mu \in \mathcal{B}^1$ is such that $A_N\tau_N\mu = T_N\mu$ . Hence

$$A_N D_N\mu = (A_N\delta_N + T_N)\mu = B_N C_N\mu. \tag{23}$$

If we use $N^k$ instead of $N$, we can define the corresponding contracted product $D_{N^k}\mu$ if only $T_{N^k}\mu \in \mathcal{B}^1_k$. If this is the case, we call such a (1, 1)-tensor field $N$ a *strong Saletan tensor* (for $\mu$). We have the following:

**Theorem 6.** *If $N$ is regular (e.g. $E$ is over a single point) and $T_N\mu$ takes values in $\mathcal{A}^1$ (i.e. $N$ is a Saletan tensor), then $N$ is a strong Saletan tensor.*

**Proof.** Indeed, in this case,

$$T_{N^k}\mu = (A^k_N - B^k_N)(A^k_N - C^k_N) = \omega(A_N, B_N, C_N)\,(A_N - B_N)(A_N - C_N)\mu$$

where $\omega$ is a polynomial and $(A_N - B_N)(A_N - C_N)\mu = T_N\mu \in \mathcal{B}^1_0$, since $N$ is a Saletan tensor. We then have $T_{N^k}\mu \in \mathcal{B}^1_0$, since $\mathcal{B}^1_0$ is an invariant subspace with respect to $A_N$, $B_N$, $C_N$. But in the regular case $N^k(\mathcal{A}^1) = \mathcal{A}^1$ (theorem 3), so $\mathcal{B}^1_0 = \mathcal{B}^1_k$. □

There is a nice algebraic condition which ensures that the tensor is regular.

**Theorem 7.** *Suppose that there is a finite-dimensional $N$-invariant subspace $V$ in $\mathcal{A}$ which generates $\mathcal{A}$ as a $C^\infty(M)$-module, i.e. the sections from $V$ span the bundle $E$. Then the tensor $N$ is regular and it is a strong Saletan tensor if and only if its Nijenhuis torsion takes values in $\mathcal{A}^1$.*

**Proof.** Let $V = V_1 \oplus V_2$ be the Riesz decomposition of $V$ with respect to $N$ (as acting on $V$). Since $N^k(V_2) = \{0\}$ for a sufficiently large $k$, $V_2 \subset \mathcal{A}_2$. Similarly, since $N(V_1) = V_1$, $V_1 \subset \mathcal{A}_1$. Since $V$ generates $E$, we have the decomposition $E(p) = V_1(p) \oplus V_2(p)$ for any $p \in M$. By the dimension argument, $V_2(p) = E_2(p)$ for every $p \in M$, so $E^2$ is a smooth distribution and its dimension is constant due to theorem 1. □

For Nijenhuis tensors we have the following.

**Theorem 8.** *If $N$ is a Nijenhuis tensor for the product $\mu$ and $w$, $v$ are polynomials, then $w(N)$ is a Nijenhuis tensor for $\delta_{v(N)}\mu$.*

**Proof.** $N$ is a Nijenhuis tensor for $\mu$, so $T_N\mu = 0$. Since $A_{w(N)} = w(A_N)$, etc, we have

$$T_{w(N)}\delta_{v(N)}\mu = \delta_{v(N)}W(A_N, B_N, C_N)\mu$$

where

$$W(x, y, z) = (w(x) - w(y))(w(x) - w(z)) = W_1(x, y, z)(x - y)(x - z)$$

for a certain polynomial $W_1$. Hence,

$$T_{w(N)}\delta_{v(N)}\mu = \delta_{v(N)} W_1(A_N, B_N, C_N)T_N\mu = 0. \qquad \square$$

For any strong Saletan tensor $N$ we get a whole hierarchy of contracted products

$$D_{N^k}\mu = \delta_{N^k}\mu + \tau_{N^k}\mu \qquad k = 1, 2, \ldots.$$

We will show that this is exactly the same hierarchy if we apply the contraction procedure inductively:

$$\mu_0 = \mu \qquad \mu_{k+1} = D_N\mu_k.$$

For the case of Nijenhuis tensors, it is very easy. Indeed, as above, $N^k$ are Nijenhuis tensors for $\mu$ for any $k = 1, 2, \ldots$ and $D_{N^k}\mu = \delta_{N^k}\mu$. To see that $\delta_{N^k}\mu = (\delta_N)^k\mu$, it is sufficient to check that

$$\left(\delta_{N^k} - (\delta_N)^k\right)\mu = \left((B_N^k + C_N^k - A_N^k) - (B_N + C_N - A_N)^k\right)\mu = 0.$$

But the polynomial $(x^k + y^k - z^k) - (x + y - z)^k$ vanishes for $x = z$ and for $y = z$, so that it can be written in the form $\omega(x, y, z)(z - x)(z - y)$. Hence,

$$\left(\delta_{N^k} - (\delta_N)^k\right)\mu = \omega(A_N, B_N, C_N)(A_N - B_N)(A_N - C_N)\mu = 0$$

since $(A_N - B_N)(A_N - C_N)\mu = 0$. For an arbitrary strong Saletan tensor the situation is a little bit more complicated. First, we show the following:

**Lemma 1.** *With the previous notation, for any strong Saletan tensor and any couple of natural numbers $i, k \in \mathbb{N}$, we have*

$$T_{N^i} D_{N^k}\mu = A_N^i(D_{N^{k+i}}\mu - \delta_{N^i} D_{N^k}\mu). \tag{24}$$

**Proof.** First of all, let us observe that both sides belong to $\mathcal{B}_0^1$. Indeed, the left-hand side equals

$$T_{N^i}(\delta_{N^k}\mu + \tau_{N^k}\mu) = \delta_{N^k}T_{N^i}\mu + T_{N^i}\tau_{N^k}\mu$$

and $T_{N^i}\mu,\ \tau_{N^k}\mu \in \mathcal{B}_0^1$, so the left-hand side also belongs to $\mathcal{B}_0^1$, due to the invariance of $\mathcal{B}_0^1$. As for the right-hand side, we write

$$A_N^i(D_{N^{k+i}}\mu - \delta_{N^i}D_{N^k}\mu) = A_N^i(\delta_{N^{k+i}}\mu - \delta_{N^i}\delta_{N^k}\mu + \tau_{N^{k+i}}\mu - \delta_{N^i}\tau_{N^k}\mu).$$

Since, similarly as above, $\tau_{N^{k+i}}\mu$ and $\delta_{N^i}\tau_{N^k}\mu$ belong to $\mathcal{B}_0^1$, it suffices to check that

$$(\delta_{N^{k+i}} - \delta_{N^i}\delta_{N^k})\,\mu \in \mathcal{B}_0^1$$

which is straightforward, since

$$\begin{aligned}(\delta_{N^{k+i}} - \delta_{N^i}\delta_{N^k})\,\mu &= (B_N^{k+i} + C_N^{k+i} - A_N^{k+i} - (B_N^i + C_N^i - A_N^i)(B_N^k + C_N^k - A_N^k))\mu \\ &= \omega(A_N, B_N, C_N)(A_N - B_N)(A_N - C_N)\mu = \omega(A_N, B_N, C_N)T_N\mu \in \mathcal{B}_0^1\end{aligned}$$

where we use an analogous polynomial factor argument as above and the invariance of $\mathcal{B}_0^1$. Hence, we can check the following by applying $A_N^k$ to both sides of (24) ($A_N$ is invertible on $E^1$):

$$A_N^k T_{N^i} D_{N^k}\mu = A_N^{i+k}(D_{N^{i+k}}\mu - \delta_{N^i} D_{N^k}\mu).$$

Writing down expressions for $D_{N^k}\mu$ and $D_{N^{k+i}}\mu$ explicitly, and using

$$A_N^k D_{N^k}\mu = (A_N^k\delta_N^k + T_{N^k})\mu = B_N^k C_N^k\mu$$

etc, we obtain

$$A_N^k T_{N^i} D_{N^k}\mu = T_{N^i} B_N^k C_N^k \mu = (B_N^i C_N^i - \delta_{N^i} A_N^i) B_N^k C_N^k \mu \tag{25}$$

$$= (B_N^{i+k} C_N^{i+k} - A_N^i \delta_{N^i} B_N^k C_N^k)\mu \tag{26}$$

$$= A_N^{i+k}(D_{N^{i+k}}\mu - \delta_{N^i} D_{N^k}\mu). \qquad \square$$

**Corollary 1.** *The tensor $N$ is a strong Saletan tensor for any of $D_{N^k}\mu$, $k = 0, 1, 2 \ldots$.*

**Theorem 9.** *If $N$ is a strong Saletan tensor for $\mu$, then:*

*(a) We have a well defined hierarchy of contracted products $D_{N^k}\mu$, $k = 0, 1, 2 \ldots$.*
*(b) $N$ is a strong Saletan tensor for every $D_{N^k}\mu$, $k = 0, 1, 2 \ldots$.*
*(c) $D_{N^i} D_{N^k}\mu = D_{N^{i+k}}\mu$, for any pair of natural numbers $i, k \in \mathbb{N}$.*
*(d) $N^k$ is a homomorphism of the product $D_{N^{i+k}}\mu$ into $D_{N^i}\mu$, for any pair of natural numbers $i, k \in \mathbb{N}$.*

**Proof.** We get (a) by definition and (b) is just the corollary above. To prove (c), let us write (24) from lemma 1 in the form

$$\tau_{N^i} D_{N^k}\mu = D_{N^{k+i}}\mu - \delta_{N^i} D_{N^k}\mu.$$

Hence,

$$D_{N^{k+i}}\mu = \delta_{N^i} D_{N^k}\mu + \tau_{N^i} D_{N^k}\mu = D_{N^i} D_{N^k}\mu.$$

Finally, (d) is straightforward. By the result of lemma 1,

$$N^i(D_{N^{k+i}}\mu(X, Y)) = (A_N^i D_{N^{k+i}}\mu)(X, Y) = (A_N^i \delta_{N^i} + T_{N^i}) D_{N^k}\mu(X, Y)$$
$$= B_N^i C_N^i D_{N^k}\mu(X, Y) = D_{N^k}\mu(N^i(X), N^i(Y)). \tag{27}$$

$\square$

**Corollary 2.** *For any strong Saletan tensor $N$ for $\mu$,*

*(a) $N^k(\mathcal{A})$ is a subalgebra with respect to the product $D_{N^i}\mu$;*
*(b) $\ker N^k = \{X \in \mathcal{A} \mid N^k(X) = 0\}$ is an ideal of $D_{N^i}\mu$, for all $i > k$.*

## 5. Behaviour of properties of algebraic structures under contraction

Assume that our product $\mu$ is a specific one, satisfying some general axioms $\{(a_\mu^i)\}$ of the form

$$(a_\mu^i) \quad \forall x_1, \ldots, x_{n_i} \in \mathcal{A} \qquad [w_\mu^i(x_1, \ldots, x_{n_i}) = 0] \tag{28}$$

where $w_\mu^i$ are $\mu$-polynomial functions, and using only universal quantifiers, like

$$(a_\mu^1) \quad \forall x, y, z, \in \mathcal{A} \qquad [\mu(x, \mu(y, z)) + \mu(y, \mu(z, x)) + \mu(z, \mu(x, y)) = 0]$$

or

$$(a_\mu^2) \quad \forall x, y \in \mathcal{A} \qquad [\mu(x, y) + \mu(y, x) = 0]$$

or

$$(a_\mu^3) \quad \forall x, y, z \in \mathcal{A} \qquad [\mu(x, \mu(y, z)) - \mu(\mu(x, y), z) = 0]$$

but not using existential quantifiers like

$$(a_\mu^4) \quad \exists 1 \in \mathcal{A} \quad \forall y \in \mathcal{A} \qquad [\mu(1, y) = y = \mu(y, 1)].$$

An algebra satisfying $(a_\mu^1)$ and $(a_\mu^2)$ is a Lie algebra, an algebra satisfying $(a_\mu^3)$ is associative, and $(a_\mu^4)$ says that $\mathcal{A}$ is unital.

**Theorem 10.** *If the product $\mu$ satisfies axioms of the form (28), then the contracted product $\mu_N$ satisfies these axioms.*

**Proof.** The products $\mu_N^\lambda = U(\lambda)^{-1} \circ \mu \circ U(\lambda)^{\otimes 2}$ are isomorphic to $\mu$, so that they satisfy the same axioms, and equations $a^i_{\mu_N^\lambda}(x_1, \ldots, x_{n_i}) = 0$ are going to $a^i_{\mu_N}(x_1, \ldots, x_{n_i}) = 0$ by passing to the limit as $\lambda \to 0$. □

**Remark.** The above theorem implies that a contraction of a Lie algebra is a Lie algebra and a contraction of an associative algebra is an associative algebra. However, it is crucial that the axioms use the universal quantifiers only. For example, the existence of unity ($a^4_\mu$) is, in general, not preserved by contractions as is shown by the case $N = 0$. This is because the unit for the product $\mu_N^\lambda$ is $U(\lambda)^{-1}(1)$ which may have no limit as $\lambda \to 0$.

**Definition 2.** *We say that products $\mu, \mu'$ satisfying axioms (28) are compatible, if any linear combination $\mu + \alpha\mu'$ satisfies these axioms. For instance, two associative products are compatible if and only if their sum is associative as well, etc.*

**Theorem 11.** *If $N$ is a Nijenhuis tensor for $\mu$, then the products $\mu$ and $\mu_N = \delta_N \mu$ are compatible.*

**Proof.** According to theorem 3, $I + \alpha N$ is a Nijenhuis tensor for $\mu$ for any $\alpha \in \mathbb{R}$. Now, using theorem 10 we see that the product

$$\mu_{(I+\alpha N)} = \delta_{(I+\alpha N)}\mu = \mu + \alpha\, \mu_N$$

satisfies the axioms of $\mu$. □

**Remark.** If $N$ is only a Saletan tensor, the products $\mu$ and $\mu_N$ are, in general, not compatible. For example, the associative products $X * Y$ and $X *_N Y = N^{-1}(N(X) * N(Y))$, for invertible $N$, are in general not compatible, i.e. $X * Y + X *_N Y$ is, in general, not associative.

## 6. Contractions of Lie algebras

Let us consider now the very important particular case of a finite-dimensional Lie algebra $(E, [\cdot,\cdot])$. This corresponds to the vector bundle $E$ over a single point with $\mathcal{A} = E$ and $\mu = [\cdot,\cdot]$. The family $U(\varepsilon)$ of endomorphisms of the underlying vector space $V$ considered by İnönü and Wigner [IW] is $U(\varepsilon) = P + \varepsilon(I - P)$, where $P$ is a projection, and it was later studied by Saletan [Sa] in the more general case,

$$U(\epsilon) = \epsilon\, I + (1 - \epsilon)N$$

for which $U(0) = N$ and $U(1) = I$. By reparametrizing it with a new parameter $\lambda = \frac{\epsilon}{1-\epsilon}$, it is, as is shown in theorem 4, equivalent to the contraction

$$[X, Y]_N = \lim_{\lambda\to 0} U(\lambda)^{-1}[U(\lambda)X, U(\lambda)Y] \tag{29}$$

with $U(\lambda) = \lambda\, I + N$. In this particular case, the Riesz decomposition $E = E^1 \oplus E^2$ with respect to $N$ is regular and, according to our general theorems 2 and 3, the necessary and sufficient condition for the existence of such a limit is that

$$T_N\mu(X, Y) = [NX, NY] - N[NX, Y] - N[X, NY] + N^2[X, Y] \in E^1. \tag{30}$$

Moreover, we obtain the following expression for the new bracket:

$$[X, Y]_N = N|_{E_1}^{-1} [NX, NY]_1 - N[X, Y]_2 + [NX, Y]_2 + [X, NY]_2$$

where the subscripts refer to the projections onto $E^1$ or $E^2$. Consequently, theorem 2 implies

$$N[X, Y]_N = [NX, NY]. \tag{31}$$

Therefore, a necessary condition for the existence of a contraction leading from a Lie algebra to become another one is the existence of a Lie algebra homomorphism of the second into the first one. However, as Levy-Nahas pointed out this is not a sufficient condition [LN].

The necessary and sufficient condition as expressed by Gilmore in [Gi]: *the contraction exists if and only if*

$$N^{p+s}[X, Y]_2 - N^p[X, N^sY]_2 = N^s[N^pX, Y]_2 - [N^pX, N^sY]_2 \qquad \text{for all} \quad p, s > 0 \tag{32}$$

can be easily obtained using techniques developed in section 4. Indeed, in the notation of section 4, equation (32) reads

$$(A_N^{p+s} - A_N^p C_N^s)\mu_2 = (A_N^s B_N^p - B_N^p C_N^s)\mu_2 \tag{33}$$

where $\mu_2$ is the projection of the bracket onto $E_2$. Since all operators commute among themselves and with the projection, we can write (33) in the form

$$\begin{aligned}(A_N^s - C_N^s)(A_N^p - B_N^p)\mu_2 &= w(A_N, B_N, C_N)((A_N - C_N)(A_N - B_N)\mu)_2 \\ &= w(A_N, B_N, C_N)(T_N\mu)_2 = 0\end{aligned}$$

which is true for all $p, s > 0$ if and only if $(T_N\mu)_2 = 0$, since the polynomial $w$ equals 1 for $p = s = 1$.

**Example 5.** Using theorem 5 we get the İnönü–Wigner contraction for Lie algebras. Consider just a splitting $E = E^1 \oplus E^2$ of the Lie algebra $E$ into a subalgebra $E^1$ and a complementary subspace $E^2$. According to theorem 5, the projection $N$ of $E$ onto $E^1$ along $E^2$ is a Saletan tensor with the splitting being also the Riesz decomposition. The resulting bracket is

$$[X, Y]_N = [X_1, Y_1] + [X_1, Y_2]_2 + [X_2, Y_1]_2. \tag{34}$$

To have a particular example, take $E = \mathfrak{su}(2)$ with the basis $X_1, X_2, X_3$ satisfying the commutation rules

$$[X_1, X_2] = X_3 \qquad [X_2, X_3] = X_1 \qquad [X_3, X_1] = X_2. \tag{35}$$

As for $E^1$, take the one-dimensional subalgebra spanned by $X_1$, and let $E^2$ be spanned by $X_2, X_3$. According to (34), the commutation rules for the contracted algebra read

$$[X_1, X_2] = X_3 \qquad [X_2, X_3] = 0 \qquad [X_3, X_1] = X_2. \tag{36}$$

One recognizes easily the Lie algebra $\mathfrak{e}(2)$ of Euclidean motions in a two-dimensional space.

As there were some contractions that could not be explained either within the framework of İnönü–Wigner [IW] or within that of Saletan [Sa], Levy-Nahas proposed a more singular contraction procedure by assuming families $U(\lambda) = \lambda^p U_s(\lambda)$, where $p \in \mathbb{N}$ and $U_s(\lambda) = N + \lambda I$. Following a quite similar path to that of Saletan contractions, one obtains as a necessary and sufficient condition for the existence of the limit in the $p = 1$ case,

$$N(T_N(X, Y)_2) = 0 \tag{37}$$

where

$$T_N(X, Y)_2 = [NX, NY]_2 - N[X, NY]_2 - N[NX, Y]_2 + N^2[X, Y]_2$$

is the projection of the Nijenhuis torsion onto $E^2$ (cf with the condition $T_N(X, Y)_2 = 0$ in the standard case). The new bracket is then $T_N(X, Y)_2$. For general $p$, the condition for the existence of the contraction is $N^p(T_N(X, Y)_2) = 0$, and the resulting bracket is $(-N)^{p-1}T_N(X, Y)_2$.

For the sake of completeness we will finally mention that other generalized İnönü–Wigner contractions were proposed in [DM, WW].

## 7. Contractions of Lie algebroids

Lie algebroids, which are very common structures in geometry, should be very nice objects for contractions in our sense, since they are, by definition, certain algebra structures on sections of vector bundles. They were introduced by Pradines [Pr] as infinitesimal objects for differentiable groupoids, but one can find similar notions proposed by several authors in an increasing number of papers (which proves their importance and naturalness). For basic properties and the literature on the subject we refer to the survey paper by Mackenzie [Mac].

**Definition 3.** *A* Lie algebroid *on a smooth manifold $M$ is a vector bundle $\tau : E \to M$, together with a bracket $\mu = [\cdot, \cdot] : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ on the $C^\infty(M)$-module $\mathcal{A} = \Gamma(E)$ of smooth sections of $\tau$, and a vector bundle morphism $a_\mu : E \to TM$, over the identity on $M$, from $E$ to the tangent bundle $TM$, called the* anchor *of the Lie algebroid, such that:*

*(a) the bracket $\mu$ is a Lie algebra bracket on $\mathcal{A}$ over $\mathbb{R}$;*
*(b) for all $X, Y \in \mathcal{A}$ and all smooth functions $f$ on $M$ we have*

$$\mu(X, fY) = f\,\mu(X, Y) + a_\mu(X)(f)\,Y; \tag{38}$$

*(c) for all $X, Y \in \mathcal{A}$,*

$$a_\mu(\mu(X, Y)) = [a_\mu(X), a_\mu(Y)] \tag{39}$$

*where the square bracket is the Lie bracket of vector fields. In other words, $a_\mu$ is a Lie algebra homomorphism.*

**Example 6.** Every finite-dimensional Lie algebra $E$ is a Lie algebroid as a bundle over a single point with the trivial anchor. More generally, any bundle of Lie algebras is a Lie algebroid with the trivial anchor.

**Example 7.** There is a canonical Lie algebroid structure on every tangent bundle $TM$ with the bracket being the standard bracket of vector fields and the anchor being just the identity map on $TM$.

**Example 8.** There is a natural Lie algebroid associated with a realization of a Lie algebra in terms of vector fields. Suppose $V$ is a Lie algebra with the bracket $[\cdot, \cdot]$ with a realization $\hat{} : V \to \mathfrak{X}(M)$ in terms of vector fields on a manifold $M$. We can view $V$ as a subspace of sections on the trivial bundle $E = M \times V$ over $M$, regarding $X \in V$ as constant sections of $E$. There is a uniquely defined Lie algebroid structure on $\mathcal{A} = \Gamma(E) = C^\infty(M, V)$ such that the Lie algebroid bracket $\mu$ and the anchor $a_\mu$ satisfy:

(a) $\mu(X, Y) = [X, Y]$ for all $X, Y \in V$;
(b) $a_\mu(X) = \hat{X}$ for every $X \in V$.

In other words, identifying $\mathcal{A}$ with $C^\infty(M) \otimes V$, the Lie algebroid bracket reads

$$\mu(f \otimes X, g \otimes Y) = fg \otimes [X, Y] + f\hat{X}(g) \otimes Y - g\hat{Y}(f) \otimes X. \tag{40}$$

**Example 9.** There is a canonical Lie algebroid structure on the cotangent bundle $T^*M$ associated with a Poisson tensor $P$ on $M$. This is the unique Lie algebroid bracket $[\cdot, \cdot]^P$ of differential 1-forms for which $[\mathrm{d}f, \mathrm{d}g]^P = \mathrm{d}\{f, g\}^P$, where $\{\cdot, \cdot\}^P$ is the Poisson bracket of functions for $P$, and the anchor map is just $P$ viewed as a bundle morphism $P : T^*M \to TM$. Explicitly,

$$[\alpha, \beta]^P = \mathcal{L}_{P(\alpha)}\beta - \mathcal{L}_{P(\beta)}\alpha - \mathrm{d}\langle P, \alpha \wedge \beta\rangle. \tag{41}$$

This Lie bracket was defined first by Fuchssteiner [Fu]. We shall comment more on this structure in the next section.

It is interesting that any contraction of a Lie algebroid bracket gives again a Lie algebroid bracket. We start with the following lemma.

**Lemma 2.** *If $\mu$ is a Lie algebroid bracket on $\mathcal{A} = \Gamma(E)$ and $a_\mu : E \to TM$ is the corresponding anchor, then for any $(1, 1)$-tensor $N$ on $E$ we have*

*(a)* $\delta_N\mu(X, fY) = f\,\delta_N(X, Y) + a_\mu(N(X))(f)\,Y$*;*
*(b)* $T_N\mu(X, fY) = f\,T_N\mu(X, Y)$

*for any $X, Y \in \mathcal{A}$, $f \in C^\infty(M)$.*

**Proof.**

(a) By definition and properties of Lie algebroid brackets,

$$\begin{aligned}\delta_N\mu(X, fY) &= \mu(N(X), fY) + \mu(X, N(fY)) - N\mu(X, fY)\\ &= f\,\mu(N(X), Y) + a_\mu(N(X))(f)\,Y + f\,\mu(X, N(Y))\\ &\quad + a_\mu(X)(f)\,N(Y) - f\,N\mu(X, Y) + a_\mu(X)(f)\,N(Y)\\ &= f\,(\mu(N(X), Y) + \mu(X, N(Y)) - N\mu(X, Y)) + a_\mu(N(X))(f)Y\\ &= f\,\delta_N\mu(X, Y) + a_\mu(N(X))(f)\,Y.\end{aligned}$$

Here we have used the fact that the multiplication by a function commutes with $N$ (i.e. $N$ is a tensor).

(b) We have

$$\begin{aligned}T_N\mu(X, fY) &= \mu(N(X), N(fY)) - N\delta_N\mu(X, fY)\\ &= f\,\mu(N(X), N(Y)) + a_\mu(N(X))(f)\,N(Y)\\ &\quad - f\,N\delta_N\mu(X, Y) - a_\mu(N(X))(f)\,N(Y)\\ &= f\,(\mu(N(X), N(Y)) - N\delta_N\mu(X, Y)) = fT_N\mu(X, Y),\end{aligned}$$

where we have used (a). □

**Theorem 12.** *If $N$ is a Saletan tensor for a Lie algebroid bracket $\mu$ on $\mathcal{A} = \Gamma(E)$, with an anchor map $a_\mu : E \to TM$, then the contracted bracket $\mu_N$ is again a Lie algebroid bracket on $\mathcal{A}$ with the anchor $a_{\mu_N} = a_\mu \circ N$.*

**Proof.** We already know that the contracted bracket $\mu_N = \delta_N\mu + \tau_N\mu$ is a Lie bracket. Since $N$ is a Saletan tensor, $\tau_N\mu = N^{-1}T_N\mu$ is well defined and clearly also satisfies (b) of the above lemma. Thus, also using (a),

$$\begin{aligned}\mu_N(X, fY) &= \delta_N\mu(X, fY) + \tau_N\mu(X, fY)\\ &= f\delta_N\mu(X, Y) + a_\mu(N(X))(f)Y + f\tau_N\mu(X, Y)\\ &= f(\delta_N\mu(X, Y) + \tau_N\mu(X, Y)) + a_\mu(N(X))(f)Y\\ &= f\mu_N(X, Y) + a_\mu(N(X))(f)Y\end{aligned}$$

so that $a_{\mu_N} = a_\mu \circ N$ can serve as the anchor of $\mu_N$. It suffices to check the condition (39):

$$\begin{aligned}[a_{\mu_N}(X), a_{\mu_N}(Y)] &= [a_\mu(N(X)), a_\mu(N(Y))]\\ &= a_\mu(\mu(N(X), N(Y)) = a_\mu(N\mu_N(X, Y)).\end{aligned}$$

We have used the identity $\mu(N(X), N(Y)) = N\mu_N(X, Y)$ which holds for Saletan tensors. □

Note that this type of contraction of Lie algebroids has already been studied by Kosmann-Schwarzbach and Magri in [KSM] in the case of Nijenhuis tensors. All the results of this section can also be applied to general algebroids as defined in [GU].

**Example 10.** The contracted bracket of vector fields defined in example 1 defines a new Lie algebroid structure on $TM$ with the anchor map being the projection onto the subbundle $E^1$ of $TM$.

**Example 11.** Any Saletan contraction of a Lie algebra $V$ leads to a contraction of the Lie algebroid associated with an action of $V$ on $M$, which was described in example 8. More precisely, if $N_0$ is a Saletan tensor for $V$, then

$$N : M \times V \to M \times V \qquad N(f \otimes X) = f \otimes N_0(X) \tag{42}$$

is a Saletan tensor for the canonical Lie algebroid bracket on $\mathcal{A} = C^\infty(M) \otimes V$. Indeed, if $V = V^1 \oplus V^2$ is the Riesz decomposition for $N_0$, then $E = E^1 \oplus E^2$, with $E^i = C^\infty(M) \otimes V^i$, is the Riesz decomposition for $N$. Moreover, the Nijenhuis torsion $T_N\mu$ takes values in $\mathcal{A}^1$. Indeed, by lemma 2(b), the Nijenhuis torsion $T_N\mu$ is tensorial, so it suffices to check that on $V$ it takes values in $\mathcal{A}^1$. But on $V$ the Nijenhuis torsion of $N$ with respect to $\mu$ is the same as the Nijenhuis torsion of $N_0$ with respect to the bracket on $V$, so it takes values in $V^1 \subset \mathcal{A}^1$. Finally, $E^1$ is of constant rank, so $N$ is regular and hence a Saletan tensor due to theorem 3. The anchor map for $\mu_N$ is $a_\mu \circ N$, so $a_{\mu_N}(f \otimes X) = f\widehat{N_0(X)}$ and the contracted anchor takes values in the module of vector fields generated by the action of the subalgebra $N_0(V)$ on $M$. In fact, what we get is the Lie algebroid structure on $M \times V$ associated with the contracted Lie algebra structure on $V$ and the anchor map $a_\mu \circ N$.

As a particular example let us take the Lie algebroid on $S^2 \times \mathfrak{su}(2)$ associated with the action of the Lie algebra $\mathfrak{su}(2)$ on the two-dimensional sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$ given by (in the notation of example 5)

$$\hat{X}_1 = y\partial_z - z\partial_y \qquad \hat{X}_2 = x\partial_z - z\partial_x \qquad \hat{X}_3 = x\partial_y - y\partial_x. \tag{43}$$

From the contraction of $\mathfrak{su}(2)$ into $\mathfrak{e}(2)$, as described in example 5, we construct a contraction of this Lie algebroid. For the Lie bracket we obtain

$$\begin{aligned}\mu_N\left(\sum f_i \otimes X_i, \sum g_j \otimes X_j\right) = {} & (f_1\hat{X}_1(g_1) - g_1\hat{X}_1(f_1)) \otimes X_1\\ & +(f_3g_1 - f_1g_3 + f_1\hat{X}_1(g_2) - g_1\hat{X}_1(f_2)) \otimes X_2\\ & +(f_1g_2 - f_2g_1 + f_1\hat{X}_1(g_3) - g_1\hat{X}_1(f_3)) \otimes X_3\end{aligned}$$

and the anchor reads

$$a_{\mu_N}\left(\sum_{i=1}^{3} f_i \otimes X_i\right) = f_1 \hat{X}_1. \tag{44}$$

This is the Lie algebroid structure on $S^2 \times \mathfrak{e}(2)$ associated with the representation $\widehat{(X_i)_N} = \delta_1^i \hat{X}_1$ of $\mathfrak{e}(2)$ in terms of vector fields on $S^2$.

## 8. Poisson contractions

Poisson brackets, being defined on functions, are brackets of sections of one-dimensional bundles and seem, at first sight, not to go under our contraction procedures. We shall show that this is not true and that our contraction method shows precisely what contraction of a Poisson tensor should be. The crucial point is that we should think about a Poisson tensor $P$ on $M$ as defining a certain Lie algebroid structure on $T^*M$ rather than defining just the Poisson bracket $\{\cdot, \cdot\}^P$ on functions. Recall from example 9 that the Lie algebroid bracket on differential forms, associated with $P$, reads

$$[\alpha, \beta]^P = \mathcal{L}_{P(\alpha)}\beta - \mathcal{L}_{P(\beta)}\alpha - \mathrm{d}\langle P, \alpha \wedge \beta\rangle. \tag{45}$$

The anchor map of this Lie algebroid is just $P$, viewed as a bundle morphism $P : T^*M \to TM$, so that $P$ can be easily decoded from the Lie algebroid structure. Of course, not all Lie algebroid structures on $T^*M$, even having a Poisson tensor for the anchor map, are of this kind. We just mention that an elegant characterization of Lie algebroid brackets associated with Poisson structures is that the exterior derivative acts as a graded derivative on the corresponding Schouten-like bracket of differential forms, or that this Lie algebroid structure constitutes a *Lie bialgebroid* (in the sense of Mackenzie and Xu [MX]) together with the canonical Lie algebroid structure on the dual, i.e. the tangent bundle $TM$.

One can think differently. Suppose we have a skew-symmetric 2-vector field $P$, viewed as a bundle morphism $P : T^*M \to TM$, and we write formally the bracket (45). When do we obtain a Lie algebroid bracket? The answer is very simple (cf [KSM]):

**Theorem 13.** *If $P$ is a skew 2-vector field, then formula (45) gives a Lie algebroid bracket if and only if $P$ is a Poisson tensor.*

Let us now fix a Poisson structure $P$ on $M$ and the Lie algebroid bracket $\mu^P = [\cdot, \cdot]^P$. Given the Saletan tensor $N$ for $\mu^P$, we obtain the contracted bracket $\mu_N^P$. It is natural, in the case when $\mu_N^P$ is again a Lie algebroid bracket associated with a Poisson tensor (we shall speak about *Poisson contraction*), to call this tensor a *contracted Poisson structure* by means of $N$. By ${}^tN$ we shall denote the dual bundle morphism of $N : T^*M \to T^*M$. In particular, ${}^tN : TM \to TM$. We almost follow the notation of [MM, KSM] but with exchanged roles for $N$ and ${}^tN$ which, as will be seen later, seems to be more appropriate in our case.

**Theorem 14.** *A necessary and sufficient condition for the contraction of the Lie algebroid $\mu^P$ associated with a Saletan tensor $N$ to be a Poisson contraction is that*

*(a)* $PN = {}^tNP$
*(b)* $\mu_N^P = \mu^{PN}$.

**Proof.** First, assume that the contraction according to $N$ is a Poisson contraction. Hence, $\mu_N^P = \mu^{P_1}$ for a Poisson tensor $P_1$. But the anchor of $\mu^{P_1}$ is $P_1$ and the anchor of $\mu_N^P$ is

$PN$ (theorem 12). We then obtain $P_1 = PN$ and (b). Since $PN$ must be skew-symmetric, ${}^t(PN) = -PN$. But ${}^t(PN) = {}^tN{}^tP = -{}^tNP$ and we obtain (a).

Suppose now (a) and (b). Since (a) means that $PN$ is skew-symmetric and $\mu_N^P$ is a Lie algebroid bracket, in view of theorem 13, the tensor $PN$ is a Poisson tensor. □

**Remark.** Bihamiltonian systems, as noticed by Magri [Mag], play an important role in the discussion of complete integrability in the sense of Liouville. A geometrical approach to these questions, proposed in [MM] (see also [KSM]), uses the notion of a *Poisson–Nijenhuis structure*, i.e. a pair $(P, N)$, where $P$ is a Poisson tensor on $M$ and $N$ is a Nijenhuis tensor on the tangent bundle $TM$, which satisfy certain compatibility conditions. For contractions of $\mu^P$, we use $N$ being a morphism of $T^*M$ rather than of $TM$, but of course, by duality, ${}^tN : TM \to TM$.

In the case when $N$ is a Nijenhuis tensor for $\mu^P$, our conditions (a) and (b) are the same as the compatibility conditions for Poisson–Nijenhuis structure of [MM, KSM] with $N$ replaced by ${}^tN$. In this case $\mu^{PN} - \mu_N^P$ is exactly what in [KSM] is denoted by $C(P, {}^tN)$. Note that Poisson–Nijenhuis structures can be described in terms of Lie bialgebroids [KS] (cf also [GU1] for a more general setting).

We do not assume that ${}^tN$ (in our notation) is a Nijenhuis tensor for the canonical Lie algebroid $TM$, but that $N$ is a Saletan tensor for $\mu^P$ on $T^*M$. It is natural to call a pair $(P, N)$, where $P$ is a Poisson structure on $M$ and $N$ is a Saletan tensor for $\mu^P$ satisfying (a) and (b) of the above theorem, a *Poisson–Saletan structure*. If $(P, N)$ is a Poisson–Saletan structure, then $(P, {}^tN)$ need not be a Poisson–Nijenhuis structure in the sense of [KSM], even if we impose that $N$ is a Nijenhuis tensor for $\mu^P$, as the following example shows. However, this weaker assumption is sufficient to perform a Poisson contraction and to obtain the contracted Poisson structure $PN$. In the case when $N$ is a Nijenhuis tensor for $\mu^P$, according to theorem 11, Lie algebroid brackets $\mu^P$ and $\mu_N^P = \mu^{PN}$ are compatible, so the Poisson tensors $P$ and $PN$ are also compatible. We can also obtain a whole hierarchy of compatible Poisson tensors using the results of section 4.

**Example 12.** Let $M = M_1 \times M_2$, where $M_i$, $i = 1, 2$, is a manifold. On the product manifold consider the product Poisson structure $P = P_1 \times \{0\}$, where $P_1$ is a Poisson structure on $M_1$. Let $N_2 : T^*M_2 \to T^*M_2$ be any $(1-1)$-tensor on $M_2$. It induces a tensor $N : T^*M \to T^*M$ which on

$$T^*_{(m_1,m_2)}M = T^*_{m_1}M_1 \oplus T^*_{m_2}M_2 \tag{46}$$

acts by identity on $T^*_{m_1}M_1$ and by $(N_2)_{m_2}$ on $T^*_{m_2}M_2$. The $C^\infty(M)$-module $\Omega^1(M)$ of 1-forms on $M$ is generated by $\Omega^1(M_1)$ and $\Omega^1(M_2)$ and, as can be easily seen from (45), $\mu^P(\alpha, \beta) = \mu^{P_1}(\alpha, \beta)$ for $\alpha, \beta \in \Omega^1(M_1)$, and $\mu^P(\alpha, \beta) = 0$ when $\alpha \in \Omega^1(M_2)$. Since $\Omega^1(M_1)$ and $\Omega^1(M_2)$ are invariant subspaces for $N$, and since $N$ acts by identity on $\Omega^1(M_1)$, it follows that $N$ is a Nijenhuis tensor for $\mu^P$ and that $\mu_N^P = \mu^P = \mu^{PN}$. Thus $(P, N)$ is a Poisson–Saletan structure. On the other hand, ${}^tN = id \times {}^tN_2$ need not to be a Nijenhuis tensor for $TM$, since $N_2$ is arbitrary.

However, we have the following weaker result.

**Theorem 15.** *If $N : T^*M \to T^*M$ is a Nijenhuis tensor for $\mu^P$, then the Nijenhuis torsion of ${}^tN : TM \to TM$ vanishes on the vector fields from the image of $P : T^*M \to TM$. In particular, if P is invertible, i.e. comes from a symplectic structure, then $(P, {}^tN)$ is a Poisson–Nijenhuis structure is the sense of [KSM].*

**Proof.** Writing down $T_N \mu^P = 0$, we obtain

$$\mu^P(N(\alpha), N(\beta)) = N(\mu^P(N(\alpha), \beta) + \mu^P(\alpha, N(\beta)) - N\mu^P(\alpha, \beta)). \tag{47}$$

Applying the anchor $P$ to both sides, we obtain, according to (39),

$$[PN(\alpha), PN(\beta)] = PN(\mu^P(N(\alpha), \beta) + \mu^P(\alpha, N(\beta)) - N\mu^P(\alpha, \beta)). \tag{48}$$

Now using $PN = {}^tNP$ and the fact that anchor is a homomorphism of the brackets once more, we obtain

$$[{}^tNP(\alpha), {}^tNP(\beta)] = {}^tN([{}^tNP(\alpha), P(\beta)] + [P(\alpha), {}^tNP(\beta)] - {}^tN[P(\alpha), P(\beta)]). \tag{49}$$

The latter means exactly that

$$T_{{}^tN}(P(\alpha), P(\beta)) = 0 \tag{50}$$

where $T_{{}^tN}$ is the Nijenhuis torsion of ${}^tN$ with respect to the bracket of vector fields. □

**Remark.** The property (50), together with the compatibility condition, defines a *weak Poisson–Nijenhuis structure* in the terminology of [MMP]. That this weak condition is sufficient to obtain recursion operators was first observed in [MN]. Note also that a similar procedure can be applied to Jacobi structures. Jacobi structures give rise to Lie algebroids as was observed in [KSB]. Similarly to above, the contraction procedures for these Lie algebroids give rise to a proper concept of a *Jacobi–Nijenhuis structure*. We refer to [MMP] for details.

## 9. Contractions of $n$-ary products and coproducts

Let, as before, $N$ be a (1, 1)-tensor over a vector bundle $E$, $E = E^1 \oplus E^2$ be the Riesz decomposition of $E$ relative to $N$, and $\mathcal{A}, \mathcal{A}^1$ be the spaces of smooth sections of $E$ and $E^1$, respectively. In complete analogy with binary products, we can consider $n$-ary products (respectively, coproducts), i.e. linear mappings $\mu : \mathcal{A}^{\otimes n} \to \mathcal{A}$ (respectively, linear mappings $\mu : \mathcal{A} \to \mathcal{A}^{\otimes n}$) and contractions of them with respect to families $U(\lambda) = \lambda I + N$. For an $n$-ary product (respectively, coproduct) we denote

$$\delta_N \mu = \mu \circ N_n^{n-1} - N \circ \mu \circ N_n^{n-2} + \cdots + (-1)^{n-1} N^{n-1} \circ \mu \circ N_n^0$$

and, respectively,

$$\delta_N \mu = N_n^{n-1} \circ \mu - N_n^{n-2} \circ \mu \circ N + \cdots + (-1)^{n-1} N_n^0 \circ \mu \circ N^{n-1}$$

where $N_n^k$ are defined by

$$(\lambda I + N)^{\otimes n} = \sum_{k=0}^{n} \lambda^{n-k} N_n^k.$$

An obvious adaptation of the proof of theorem 2 gives the following.

**Theorem 16.** *Let $\mu : \mathcal{A}^{\otimes n} \to \mathcal{A}$ be a pointwise continuous $n$-ary product in $\mathcal{A}$. Then, for $U(\lambda) = \lambda I + N$, the limit*

$$\mu_N = \lim_{\lambda \to 0} (U(\lambda)^{-1} \circ \mu \circ U(\lambda)^{\otimes n})$$

*exists and defines a new (contracted) $n$-ary product $D_N\mu$ on $\mathcal{A}$ if and only if the Nijenhuis torsion*

$$T_N \mu = \mu \circ N^{\otimes n} - N \circ \delta_N \mu$$

*takes values in* $N(\mathcal{A}^1)$*. If this is the case, then*

$$\mu_N = \delta_N\mu + \tau_N\mu \tag{51}$$

*where* $N(\tau_N\mu) = T_N\mu$*. Moreover,* $N$ *constitutes a homomorphism of* $(\mathcal{A}, \mu_N)$ *into* $(\mathcal{A}, \mu)$*:*

$$N \circ \mu_N = \mu \circ N^{\otimes n}.$$

A similar theorem for coproducts can be obtained by duality. Since it is much harder to put conditions for existence of contraction on arguments of the Nijenhuis torsion, for simplicity we give an explicit version for the regular case only.

**Theorem 17.** *Let* $\mu : \mathcal{A} \to \mathcal{A}^{\otimes n}$ *be a pointwise continuous* $n$*-ary coproduct in* $\mathcal{A}$*. If* $N$ *is regular, i.e.* $E^1$ *is of constant dimension, then, for* $U(\lambda) = \lambda I + N$*, the limit*

$$\mu_N = \lim_{\lambda\to 0}(U(\lambda)^{\otimes n} \circ \mu \circ U(\lambda)^{-1})$$

*exists and defines a new (contracted)* $n$*-ary coproduct* $D_N\mu$ *on* $\mathcal{A}$ *if and only if the Nijenhuis torsion*

$$T_N\mu = N^{\otimes n} \circ \mu - \delta_N\mu \circ N$$

*vanishes on* $\mathcal{A}^2$*-the space of sections of* $E^2$*. If this is the case, then*

$$\mu_N = \delta_N\mu + \tau_N\mu \tag{52}$$

*where* $(\tau_N\mu) = T_N\mu \circ N^{-1}$ *on* $\mathcal{A}^1$ *and* $\tau_N\mu = 0$ *on* $\mathcal{A}^2$*. Moreover,* $N$ *constitutes a homomorphism of* $(\mathcal{A}, \mu_N)$ *into* $(\mathcal{A}, \mu)$*:*

$$\mu_N \circ N = N^{\otimes n} \circ \mu.$$

One can consider more general algebraic structures of the form $\mu : \mathcal{A}^{\otimes k} \to \mathcal{A}^{\otimes n}$, but this leads to more conditions of contractibility and we will not study these cases in the present paper. Note only that contractions of coproducts, as a part of contractions of Lie bialgebras, appeared already in [BGHOS].

## 10. Conclusions

Motivated by physical examples from quantum mechanics, we have studied contractions of general binary (or $n$-ary) products with respect to one-parameter families of transformations of the form $U(\lambda) = \lambda A + N$, generalizing pioneering work by İnönü, Wigner and Saletan. Our generalization can be applied to many infinite-dimensional cases, especially Lie algebroids and Poisson brackets, however, it does not deal with a generic dependence of the contraction parameter. The problem of describing contractions with respect to general $U(\lambda)$, or even differentiable with respect to $\lambda$, is much more complicated.

The contraction procedure can be viewed as an inverse of a deformation procedure. Deformations of associative and Lie algebras, at least on the infinitesimal level, are related to some cohomology. It would be interesting to relate formally deformations to contractions and also connect the cohomology to contractions.

We postpone these problems to a separate paper.

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